

DENSITY ESTIMATES FOR A VARIATIONAL MODEL DRIVEN BY THE GAGLIARDO NORM

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ABSTRACT. We prove density estimates for level sets of minimizers of the energy

$$\varepsilon^{2s} \|u\|_{H^s(\Omega)}^2 + \int_{\Omega} W(u) dx,$$

with $s \in (0, 1)$, where $\|u\|_{H^s(\Omega)}$ denotes the total contribution from Ω in the H^s norm of u , and W is a double-well potential.

As a consequence we obtain, as $\varepsilon \rightarrow 0^+$, the uniform convergence of the level sets of u to either a H^s -nonlocal minimal surface if $s \in (0, \frac{1}{2})$, or to a classical minimal surface if $s \in [\frac{1}{2}, 1)$.

1. INTRODUCTION

A classical model for the energy of a two-phase fluid of density u lying in a bounded domain $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is given by the Ginzburg-Landau energy functional

$$\int_{\Omega} \frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) dx.$$

The function $W : \mathbb{R} \rightarrow [0, +\infty)$ is a double well potential with two zeros (minima) at the densities of the stable phases, which we assume for simplicity to be $+1$ and -1 . The kinetic energy is given by the Dirichlet integral

$$\frac{\varepsilon^2}{2} \int |\nabla u|^2 dx$$

which takes into account interactions at small scales between the fluid particles. The typical energy minimizer has two regions where u is close to $+1$ and -1 which are separated by a “phase transition” which lies in an ε neighborhood of the 0 level set $\{u = 0\}$.

In this paper we consider a different model in which the kinetic term is replaced by the H^s (semi)norm of u , i.e.

$$\varepsilon^{2s} \|u\|_{H^s}^2 \quad \text{with } s \in (0, 1).$$

This means that the interactions at small scales have nonlocal character. In this case the boundary data for u is defined in $\mathcal{C}\Omega$, that is the complement of Ω . Similar models driven by a fractional, Gagliardo-type norm were considered in [12, 13]; see also [1, 14, 15] and references therein for a onedimensional related system that models phase transitions on an interval.

OS has been supported by NSG grant 0701037. EV has been supported by FIRB project “Analysis and Beyond” and GNAMPA project “Equazioni nonlineari su varietà: proprietà qualitative e classificazione delle soluzioni”. Part of this work was carried out while EV was visiting Columbia University.

From the physical point of view, the importance of these type of models relies in their attempt to capture, via the nonlocal term, the features arising from the long-range particle interactions, and it is of course desirable to understand if and how the nonlocal aspect influences the interfaces and to have good estimates on their width. Our results are a first attempt to give some answers to this questions. Indeed, we show that the level sets of the minimizers for this nonlocal energy satisfy a uniform density property. For the Allen-Cahn-Ginzburg-Landau energy such density estimates were proved in [7]. As a consequence we obtain that when $s \in (0, 1/2)$ the phase transition converges locally uniformly as $\varepsilon \rightarrow 0^+$ to a H^s -nonlocal minimal surface (see [8] for the precise definition), and when $s \in [1/2, 1)$ the phase transition converges locally uniformly to a classical minimal surface.

We define

$$X := \{u \in L^\infty(\mathbb{R}^n) \text{ s.t. } \|u\|_{L^\infty(\mathbb{R}^n)} \leq 1\},$$

the space of admissible functions – when dealing with a minimization problem in Ω , we prescribe $u \in X$ with boundary data u_o outside Ω (i.e., $u = u_o$ in $\mathcal{C}\Omega$), and we say that a sequence $u_n \in X$ converges to u in X if u_n converges to u in $L^1(\Omega)$.

We define also

$$\mathcal{K}(u; \Omega) := \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} \int_{\mathcal{C}\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy,$$

the Ω contribution in the H^s norm of u

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy,$$

i.e we omit the set where $(x, y) \in \mathcal{C}\Omega \times \mathcal{C}\Omega$ since all $u \in X$ are fixed outside Ω .

The energy functional J_ε in Ω is defined as

$$J_\varepsilon(u; \Omega) := \varepsilon^{2s} \mathcal{K}(u; \Omega) + \int_{\Omega} W(u) dx.$$

Throughout the paper we assume that $W : [-1, 1] \rightarrow [0, \infty)$,

$$(1.1) \quad \begin{aligned} W &\in C^2([-1, 1]), \quad W(\pm 1) = 0, \quad W > 0 \quad \text{in } (-1, 1) \\ W'(\pm 1) &= 0, \quad \text{and} \quad W''(\pm 1) > 0. \end{aligned}$$

We say that u is a minimizer¹ for J_ε in Ω if

$$J_\varepsilon(u; \Omega) \leq J_\varepsilon(v; \Omega)$$

for any v which coincides with u in $\mathcal{C}\Omega$.

We remark is that if u minimizes J_ε in Ω then it minimizes J_ε in any subdomain $\Omega' \subset \Omega$ since

$$\begin{aligned} \mathcal{K}(u; \Omega) &= \mathcal{K}(u; \Omega') \\ &+ \frac{1}{2} \int_{\Omega - \Omega'} \int_{\Omega - \Omega'} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega - \Omega'} \int_{\mathcal{C}\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy, \end{aligned}$$

and the latter two integral terms do not depend on the values of u in Ω' .

If u is a minimizer for J_ε in all bounded open sets Ω , we say, simply, that u is a minimizer². The behavior of J_ε as $\varepsilon \rightarrow 0^+$ is quite different as $s \in (0, 1/2)$, $s = 1/2$ and $s \in (1/2, 1)$. We showed in [21] that in each case J_ε must be multiplied by

¹A similar notion of minimizer will hold, later on, for the suitably rescaled versions of J_ε , namely \mathcal{F}_ε and \mathcal{E} .

²Sometimes, in the literature, minimizers are called “local”, or “class A”, minimizers.

an appropriate constant depending on ε in order to obtain the Γ -convergence to a limiting functional. More precisely, given any $\varepsilon > 0$, we define the functional $\mathcal{F}_\varepsilon : X \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$(1.2) \quad \mathcal{F}_\varepsilon(u) = \mathcal{F}_\varepsilon(u; \Omega) := \begin{cases} \varepsilon^{-2s} J_\varepsilon(u; \Omega) & \text{if } s \in (0, 1/2), \\ |\varepsilon \log \varepsilon|^{-1} J_\varepsilon(u; \Omega) & \text{if } s = 1/2, \\ \varepsilon^{-1} J_\varepsilon(u; \Omega) & \text{if } s \in (1/2, 1). \end{cases}$$

In the case when $s \in (0, 1/2)$, the limiting functional $\mathcal{F} : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as

$$(1.3) \quad \mathcal{F}(u) := \begin{cases} \mathcal{H}(u; \Omega) & \text{if } u|_\Omega = \chi_E - \chi_{\mathcal{C}E}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

In this case, \mathcal{F} agrees with the nonlocal area functional of ∂E in Ω that was studied in [8, 9, 2]. Remarkably, such nonlocal area functional is well defined exactly when $s \in (0, 1/2)$.

In the case when $s \in [1/2, 1)$ the limiting functional $\mathcal{F} : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as

$$(1.4) \quad \mathcal{F}(u) := \begin{cases} c_\star \text{Per}(E; \Omega) & \text{if } u|_\Omega = \chi_E - \chi_{\mathcal{C}E}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

where c_\star is a constant depending on n , s and W .

We recall the Γ -convergence results in [21]:

Theorem 1.1. *Let $s \in (0, 1)$ and Ω be a Lipschitz domain. Then, \mathcal{F}_ε Γ -converges to \mathcal{F} , i.e., for any $u \in X$,*

(i) *for any u_ε converging to u in X ,*

$$\mathcal{F}(u) \leq \liminf_{\varepsilon \rightarrow 0^+} \mathcal{F}_\varepsilon(u_\varepsilon),$$

(ii) *there exists u_ε converging to u in X such that*

$$\mathcal{F}(u) \geq \limsup_{\varepsilon \rightarrow 0^+} \mathcal{F}_\varepsilon(u_\varepsilon).$$

Theorem 1.2. *If $\mathcal{F}_\varepsilon(u_\varepsilon; \Omega)$ is uniformly bounded for a sequence of $\varepsilon \rightarrow 0^+$, then there exists a convergent subsequence*

$$u_\varepsilon \rightarrow u_\star := \chi_E - \chi_{\mathcal{C}E} \quad \text{in } L^1(\Omega).$$

Moreover, if u_ε minimizes \mathcal{F}_ε in Ω ,

- (i) *if $s \in (0, \frac{1}{2})$ and u_ε converges weakly to u_o in $\mathcal{C}\Omega$, then u_\star minimizes \mathcal{F} in (1.3) among all the functions that coincide with u_o in $\mathcal{C}\Omega$;*
- (ii) *if $s \in [1/2, 1)$, then u_\star minimizes \mathcal{F} in (1.4).*

Theorem 1.1 may be seen as a nonlocal analogue of the celebrated Γ -convergence result of [17] (see also [16, 3, 18] for further extensions). In this framework, we recall that a very important issue, besides Γ -convergence, is the “geometric” convergence of the level sets of minimizers to the limit surface. This topic has been widely studied in the case of local functionals by using appropriate density estimates (see [7], and also [11] and references therein for several other applications). The idea of these density estimates is to give an optimal bound on the measure occupied by the level sets of a minimizer in a ball.

Our results give a nonlocal counterpart of these density estimates for minimizers of J_ε (or \mathcal{F}_ε). For this, it is convenient to scale space by a factor of ε^{-1} so that

the dependence of J_ε on ε disappears. To be more precise, if u minimizes J_ε in Ω , then the rescaled function

$$u_\varepsilon(x) := u(\varepsilon x)$$

minimizes \mathcal{E} in $\Omega_\varepsilon := \Omega/\varepsilon$, where

$$\mathcal{E}(v; \tilde{\Omega}) := J_1(v; \tilde{\Omega}) = \mathcal{K}(v; \tilde{\Omega}) + \int_{\tilde{\Omega}} W(v) dx.$$

Our first result gives a uniform bound for the energy \mathcal{E} of a minimizer in B_R for large R .

Theorem 1.3. *Let u be a minimizer of \mathcal{E} in B_{R+2} with $R \geq 1$. Then*

$$(1.5) \quad \mathcal{E}(u; B_R) \leq \begin{cases} \overline{C} R^{n-2s} & \text{if } s \in (0, 1/2), \\ \overline{C} R^{n-1} \log R & \text{if } s = 1/2, \\ \overline{C} R^{n-1} & \text{if } s \in (1/2, 1), \end{cases}$$

where \overline{C} is a positive constant depending on n , s , and W .

Theorem 1.3 can be stated in terms of minimizers u_ε of \mathcal{F}_ε in $B_{1+2\varepsilon}$ as

$$\mathcal{F}_\varepsilon(u_\varepsilon; B_1) \leq \overline{C}.$$

Then, we have the following density estimate on the level sets of minimizers:

Theorem 1.4. *Let u be a minimizer of \mathcal{E} in B_R . Then for any $\theta_1, \theta_2 \in (-1, 1)$ such that*

$$(1.6) \quad u(0) > \theta_1,$$

we have that

$$(1.7) \quad |\{u > \theta_2\} \cap B_R| \geq \bar{c} R^n$$

if $R \geq \overline{R}(\theta_1, \theta_2)$. The constant $\bar{c} > 0$ depends only on n , s and W and $\overline{R}(\theta_1, \theta_2)$ is a large constant that depends also on θ_1 and θ_2 .

By scaling, Theorem 1.4 gives the uniform density estimate for minimizers u_ε of \mathcal{F}_ε (or J_ε) in B_r : if $u_\varepsilon(0) > \theta_1$ then

$$|\{u_\varepsilon > \theta_2\} \cap B_r| \geq \bar{c} r^n \quad \text{if } r \geq \overline{R}\varepsilon.$$

Remark 1.5. Our assumptions on W are not the most general. For example in the Theorem 1.3 it suffices to say that W is bounded and $W(\pm 1) = 0$. Also in Theorem 1.4 it suffices to assume that there exists a small constant $c > 0$ such that

$$(1.8) \quad \begin{aligned} &W(t) \geq W(r) + c(1+r)(t-r) + c(t-r)^2 \text{ when } -1 \leq r \leq t \leq -1+c \\ &\text{and } W(r) - W(t) \leq (1+r)/c \text{ when } -1 \leq r \leq t \leq +1. \end{aligned}$$

Of course, (1.8) is warranted by our assumptions in (1.1), but we would like to stress that less smooth, or even discontinuous, potentials, may be dealt with using (1.8).

The proof of Theorem 1.4 is contained in Section 3 and it requires a careful analysis of the measure theoretic properties of the minimizers and several nontrivial modifications of the original proof of [7], together with some iteration techniques of [8]. In particular, the construction of a new barrier function is needed in order to keep track of the densities of the level sets in larger and larger balls. Also, the

proof of (1.7) is somewhat delicate and it requires³ the following estimate for the double integral

$$(1.9) \quad L(A, D) := \int_A \int_D \frac{1}{|x - y|^{n+2s}} dx dy.$$

Theorem 1.6. *Let $s \in (0, 1)$. Let A and B be disjoint measurable subsets of \mathbb{R}^n and let $D := \mathcal{C}(A \cup B)$. Then, there exists $c \in (0, 1)$, possibly depending on n and s , for which the following estimates hold:*

- if $|B| \leq c|A|$ and $|A| > 0$, then

$$(1.10) \quad L(A, D) \geq \begin{cases} c|A|^{(n-2s)/n} & \text{if } s \in (0, 1/2), \\ c|A|^{(n-1)/n} \log(|A|/|B|) & \text{if } s = 1/2, \\ c|A|^{(n-2s)/n} (|B|/|A|)^{1-2s} & \text{if } s \in (1/2, 1), \end{cases}$$

- if $|B| > c|A|$ then

$$(1.11) \quad L(A, D) \geq c|A|^{(n-2s)/n} (|B|/|A|)^{-2s/n}.$$

Theorems 1.3 and 1.4 have, of course, physical relevance, since they give optimal bounds on the energy of the limit interface, and on the measure of the level sets of the minimizers – i.e., roughly speaking, on the probability of finding a given phase in a certain portion of the medium.

Also, due to the work of [7], density estimates as the ones in Theorem 1.4 are known to have useful scaling properties and to play a crucial role in the geometric analysis of the level sets of the rescaled minimizers, especially in relation with the asymptotic interface. For instance, we point out the following consequence of Theorem 1.1 and Theorem 1.4.

Corollary 1.7. *Suppose that u is a global minimizer of \mathcal{E} in \mathbb{R}^n , i.e. minimizes \mathcal{E} in any bounded domain $\Omega \subset \mathbb{R}^n$. Let*

$$u_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right).$$

Then

- (i) u_ε is a global minimizer for \mathcal{F}_ε ,
- (ii) u_ε converges, up to subsequences, in $L^1_{\text{loc}}(\mathbb{R}^n)$ to some $u_\star = \chi_E - \chi_{\mathcal{C}E}$ and u_\star is a global minimizer of \mathcal{F} (see (1.3), (1.4)),
- (iii) given any $\theta \in (0, 1)$, the set $\{|u_\varepsilon| \leq \theta\}$ converges to ∂E locally uniformly, that is, for any $R > 0$ and any $\delta > 0$ there exists $\varepsilon_o \in (0, 1]$, possibly depending on R and δ , such that, if $\varepsilon \in (0, \varepsilon_o]$ then

$$(1.12) \quad \{|u_\varepsilon| \leq \theta\} \cap B_R \subseteq \bigcup_{p \in \partial E} B_\delta(p).$$

The minimizer u above satisfies the Euler-Lagrange equation

$$(1.13) \quad (-\Delta)^s u(x) + W'(u(x)) = 0,$$

where

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy$$

³When $s \in (0, 1/2)$ there is also an alternative approach based on the fractional Sobolev inequality. We will perform this different proof in [22].

and the integral is understood in the principal value sense. As usual, $(-\Delta)^s$ is (up to a normalizing multiplicative constant, depending on n and s) the fractional power of the positive operator $-\Delta$.

Corollary 1.7 follows immediately: (i) from the scaling properties of \mathcal{E} , (ii) from Theorem 1.2, and (iii) is a consequence of the density estimates for the level sets of u_ε and the L^1_{loc} -convergence to u_\star (see [7] for further details).

The minimizing property of u_\star of Corollary 1.7(ii) says that when $s \in (0, 1/2)$ the limit interface ∂E is a nonlocal minimal surface in the setting of [8], and when $s \in [1/2, 1)$, ∂E is a classical minimal surface. This is interesting also because any regularity or rigidity property proved for ∂E may reflect into similar ones for the minimizers of \mathcal{F}_ε (see, e.g., [20]). In particular, (1.13) may be seen as a semilinear equation driven by the fractional Laplacian. Some rigidity properties for this kind of equations have been recently obtained, for instance, in [6, 23], but many fundamental questions on this subject are still open.

The paper is organized as follows. In Section 2 we prove Theorem 1.3. In Section 3 we prove Theorem 1.4 by treating the cases $s \in (0, 1/2)$ and $s \in [1/2, 1)$ separately. Theorem 1.6, together with a localized version of it (i.e., Proposition 4.3), is proved in Section 4. Often in the proofs, when there is no possibility of confusion, we denote the constants by C and c although they may change from line to line.

2. PROOF OF THEOREM 1.3

We use the following notation

$$u(A, B) = \int_A \int_B \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy.$$

Since

$$\begin{aligned} \mathcal{E}(u, B_R) &= \mathcal{K}(u, B_R) + \int_{B_R} W(u) dx \\ &\leq \frac{1}{2} u(B_{R+1}, B_{R+1}) + u(B_R, \mathcal{C}B_{R+1}) + \int_{B_R} W(u) dx, \end{aligned}$$

it suffices to bound each term on the right by the quantity that appears in (1.5).

We define

$$(2.1) \quad \psi(x) = -1 + 2 \min\{|x| - R - 1, 1\}$$

so that $\psi = -1$ in B_{R+1} and $\psi = 1$ in $\mathcal{C}B_{R+2}$.

First we show that $\mathcal{E}(\psi, B_{R+2})$ satisfies the bounds in (1.5). Let

$$d(x) := \max\{R - |x|, 1\}$$

and notice that

$$|\psi(x) - \psi(y)| \leq \begin{cases} 2d(x)^{-1}|x - y| & \text{if } |x - y| < d(x), \\ 2 & \text{if } |x - y| \geq d(x). \end{cases}$$

We obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{n+2s}} dy \\ &\leq \omega_{n-1} \int_0^{d(x)} \frac{(2r/d(x))^2}{r^{n+2s}} r^{n-1} dr + \omega_{n-1} \int_{d(x)}^\infty \frac{4}{r^{n+2s}} r^{n-1} dr \leq Cd(x)^{-2s}. \end{aligned}$$

Now we integrate this inequality for all $x \in B_{R+2}$ and obtain that $\mathcal{K}(\psi, B_{R+2})$ (therefore $\mathcal{E}(\psi, B_{R+2})$) satisfies the energy bound of the theorem.

Let $v = \min\{u, \psi\}$ and denote $A := \{v \leq u\} \cap B_{R+2}$. Clearly $B_{R+1} \subset A$ and $u = v$ in $\mathcal{C}A$. We write that u is a minimizer for \mathcal{E} in B_{R+2} , and therefore in A :

$$(2.2) \quad \begin{aligned} & \frac{1}{2}u(A, A) + u(A, \mathcal{C}A) + \int_A W(u) dx \\ & \leq \frac{1}{2}v(A, A) + v(A, \mathcal{C}A) + \int_A W(v) dx. \end{aligned}$$

If $x \in A$ and $y \in \mathcal{C}A$ then $v(x) = \psi(x) \leq u(x)$, $v(y) = u(y) \leq \psi(y)$ thus

$$|v(x) - v(y)| \leq \max\{|u(x) - u(y)|, |\psi(x) - \psi(y)|\},$$

which gives

$$v(A, \mathcal{C}A) \leq u(A, \mathcal{C}A) + \psi(A, \mathcal{C}A).$$

We use this in the energy inequality (2.2), we simplify $u(A, \mathcal{C}A)$ on both sides, and we obtain

$$\begin{aligned} & \frac{1}{2}u(A, A) + \int_A W(u) dx \\ & \leq \frac{1}{2}\psi(A, A) + \psi(A, \mathcal{C}A) + \int_A W(v) dx = \mathcal{E}(\psi, A) \leq \mathcal{E}(\psi, B_{R+2}). \end{aligned}$$

Since $B_{R+1} \subset A$ we obtain the desired bounds for $u(B_{R+1}, B_{R+1})$ and $\int_{B_R} W(u) dx$.

On the other hand $u(B_R, \mathcal{C}B_{R+1})$ also satisfies a similar bound since

$$\int_{\mathcal{C}B_{R+1}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy \leq C \int_{d(x)}^{\infty} r^{-1-2s} dr \leq Cd(x)^{-2s} \quad \text{for all } x \in B_R,$$

and then we integrate in $x \in B_R$.

3. PROOF OF THEOREM 1.4

3.1. Preliminary computations. A minimizer u of \mathcal{E} in B_R with $R \geq 2$ satisfies the Euler-Lagrange equation in (1.13), hence,

$$\|u\|_{C^\alpha(B_1)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|W'\|_{\mathbb{R}^n}) \leq C,$$

for some small $\alpha > 0$.

This shows that, by relabeling θ_1 , we can replace (1.6) by

$$(3.1) \quad |\{u > \theta_1\} \cap B_{R_o}| \geq \mu$$

for some constants $R_o > 0$ and $\mu > 0$, possibly depending on n, s and W – and, in fact, the proof of Theorem 1.4 will make use only of (3.1) rather than (1.6).

The strategy of the proof is, roughly speaking, to use the minimality of u in order to obtain an estimate of $|\{u > \theta_2\} \cap B_{2\rho}|$ in terms of $|\{u > \theta_2\} \cap B_\rho|$. Then the conclusion will follow by iterating (3.1).

First, we construct the following useful barrier:

Lemma 3.1. *Let $n \geq 1$. Given any $\tau > 0$, there exists $C \geq 1$, possibly depending on n, s and τ , such that the following holds: for any $R \geq C$, there exists a rotationally symmetric function*

$$(3.2) \quad w \in C(\mathbb{R}^n, [-1 + CR^{-2s}, 1]),$$

with

$$(3.3) \quad w = 1 \text{ in } \mathcal{C}B_R,$$

such that

$$(3.4) \quad -(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{w(y) - w(x)}{|x - y|^{n+2s}} dy \leq \tau(1 + w(x))$$

and

$$(3.5) \quad \frac{1}{C}(R + 1 - |x|)^{-2s} \leq 1 + w(x) \leq C(R + 1 - |x|)^{-2s}$$

for any $x \in B_R$.

Proof. We fix a large $r \geq 1$, to be conveniently chosen with respect to R and τ in the sequel.

For $t \in (0, +\infty)$ and $x \in \mathbb{R}^n$, we define

$$\begin{aligned} g(t) &:= t^{-2s}, \\ h(t) &:= \begin{cases} \min \left\{ 1, g(t) - g(r/2) - g'(r/2)(t - (r/2)) \right\} & \text{if } t \leq r/2, \\ 0 & \text{if } t \geq r/2, \end{cases} \\ v(x) &:= \begin{cases} h(r - |x|) & \text{if } x \in B_r, \\ 1 & \text{if } x \in \mathcal{C}B_r. \end{cases} \end{aligned}$$

Such a v , up to a proper rescaling, will provide the existence of the desired function w . To check this, we first notice that

$$\begin{aligned} &\text{if } t \leq r/2 \text{ and } h(t) < 1 \text{ then} \\ (3.6) \quad &h(t) = g(t) - g(r/2) - g'(r/2)(t - (r/2)) \\ &\geq g(t) - g(r/2) - |g'(r/2)|(r/2) \\ &\geq t^{-2s} - 16r^{-2s}. \end{aligned}$$

Moreover, v is continuous, radially symmetric, radially nondecreasing and $0 \leq v \leq 1$, due to the convexity of g .

Also, we claim that

$$(3.7) \quad \text{for any } x \in B_r, \|D^2 v\|_{L^\infty(B_{(r-|x|)/2}(x))} \leq 2^8(r - |x|)^{-2(1+s)}.$$

To prove (3.7), we observe that $v = 0$ in $B_{r/2}$ and so $D^2 v = 0$ in $B_{r/2}$. Then, we take $y \in B_{(r-|x|)/2}(x) \cap (\mathcal{C}B_{r/2})$ and we observe that

$$|y| \leq |y - x| + |x| \leq \frac{r - |x|}{2} + |x| = \frac{r + |x|}{2},$$

hence

$$r - |y| \geq r - \frac{r + |x|}{2} = \frac{r - |x|}{2}.$$

In particular,

$$|r - |y|| = r - |y| \leq |y|$$

and, as a consequence,

$$\begin{aligned} |D^2 v(y)| &\leq \max \left\{ |g''(r - |y|)|, \frac{|g'(r - |y|)|}{|r - |y||} \right\} \\ &\leq 2s(1 + 2s)(r - |y|)^{-2(1+s)} \leq \frac{2^8}{(r - |x|)^{-2(1+s)}}, \end{aligned}$$

proving (3.7).

From Lemma 6.15 in [19] and (3.7), we obtain that

$$\begin{aligned}
 & \text{for any } x \in B_r, \\
 & \left| \int_{\mathbb{R}^n} \frac{v(y) - v(x)}{|x - y|^{n+2s}} dy \right| \\
 (3.8) \quad & \leq C_0 \left[\|D^2 v\|_{L^\infty(B_{(r-|x|)/2}(x))} \left((r - |x|/2)^{2(1-s)} + 2 \left((r - |x|/2)^{-2s} \right) \right] \right. \\
 & \leq C_1 \left[(r - |x|)^{-4s} + (r - |x|)^{-2s} \right] \\
 & \leq C_2 (r - |x|)^{-2s},
 \end{aligned}$$

for suitable $C_0, C_1, C_2 > 0$.

Now, we claim that

$$(3.9) \quad \{v < 1\} \subseteq B_{r-(1/2)}.$$

Indeed, we take $x \in \{v < 1\}$ and we define $t_x := r - |x|$, so $h(t_x) < 1$. Hence either $t_x > r/2$ or $0 < t_x \leq r/2$ with $h(t_x) < 1$. In the first case, we would have that $|x| \leq r/2 < r - 1$ if r is large enough and (3.9) would hold, therefore we focus on the second case. But then, recalling (3.6), for large r ,

$$1 > t_x^{-2s} - 16r^{-2s} \geq t_x^{-2s} - (2^{2s} - 1).$$

That is, $t_x \geq 1/2$, proving (3.9).

A straightforward consequence of (3.9) is that

$$(3.10) \quad \|D^2 v\|_{L^\infty(\{v < 1\})} \leq C_3,$$

for a suitable $C_3 > 0$.

Now, we set

$$\Xi(x) := \begin{cases} \nabla v(x) & \text{if } x \in \{v < 1\}, \\ 0 & \text{if } x \in \{v = 1\} \end{cases}$$

and we claim that

$$(3.11) \quad v(y) - v(x) - \Xi(x) \cdot (y - x) \leq C_3 |x - y|^2,$$

for any $x \in B_r$ and any $y \in \mathbb{R}^n$.

To prove the claim above, we observe that, if $x \in \{v = 1\}$, then the left hand side of (3.11) is nonpositive, so (3.11) holds true. Also, (3.11) follows from (3.10) if both x and y lie in $\{v < 1\}$, so it only remains to prove (3.11) when $v(x) < 1$ and $v(y) = 1$. In such a case, we define v^\sharp to be a smooth, radial extension of v outside $\{v < 1\}$ such that $1 \leq v^\sharp \leq 2$ outside $\{v < 1\}$ and $\|D^2 v^\sharp\|_{L^\infty(\{v < 1\})} \leq C_3$. Then,

$$\begin{aligned}
 v(y) - v(x) - \Xi(x) \cdot (y - x) &= 1 - v^\sharp(x) - \nabla v^\sharp(x) \cdot (y - x) \\
 &\leq v^\sharp(y) - v^\sharp(x) - \nabla v^\sharp(x) \cdot (y - x) \leq C_3 |x - y|^2,
 \end{aligned}$$

proving (3.11) in this case too.

Thus, thanks to (3.11), we may use estimate (6.47) in Lemma 6.14 of [19] and obtain that

$$(3.12) \quad \text{for any } x \in B_r, \quad \int_{\mathbb{R}^n} \frac{v(y) - v(x)}{|x - y|^{n+2s}} dy \leq C_4$$

for a suitable $C_4 > 0$.

Now, we point out that

$$(3.13) \quad \min \{1, t^{-2s}\} \leq h(t) + 16r^{-2s}.$$

Indeed, if $t \leq r/2$, then (3.13) is a consequence of (3.6), while if $t > r/2$ we have that $t^{-2s} \leq 8r^{-2s} < 1$, which implies (3.13).

As a consequence of (3.13), we have that

$$(3.14) \quad \min \{1, (r - |x|)^{-2s}\} \leq v(x) + 16r^{-2s}.$$

From (3.8), (3.12) and (3.14), we conclude that

$$(3.15) \quad \begin{aligned} \text{for any } x \in B_r, \quad & \int_{\mathbb{R}^n} \frac{v(y) - v(x)}{|x - y|^{n+2s}} dy \\ & \leq C_5 \min \{1, (r - |x|)^{-2s}\} \\ & \leq C_5 (v(x) + 16r^{-2s}), \end{aligned}$$

for a suitable $C_5 > 0$.

Moreover, for any $t \in [0, r/2]$,

$$h(t) \leq g(t) + |g'(r/2)| (r/2) \leq t^{-2s} + 8r^{-2s}$$

and so

$$(3.16) \quad \text{for any } x \in B_r, v(x) \leq (r - |x|)^{-2s} + 8r^{-2s}.$$

Now we define

$$\begin{aligned} C_o &:= \left(\frac{C_5}{\tau} \right)^{1/2s}, \\ \beta &:= 32r^{-2s} \\ w(x) &:= (2 - \beta)v\left(\frac{x}{C_o}\right) + \beta - 1. \end{aligned}$$

and we take $r := R/C_o$. Notice that r is large if so is R , possibly in dependence of τ (thus, from now on, the constants are also allowed to depend on τ). Also, w is radially non decreasing, and $w = 1$ in $\mathcal{C}B_R$. In particular,

$$(3.17) \quad 1 + w(x) \leq 2.$$

Moreover,

$$(3.18) \quad w(x) = \beta - 1 \text{ for any } x \in B_{R/2}$$

and, from (3.16), for any $x \in B_R$,

$$1 + w(x) \leq 2v(x/C_o) + \beta \leq 2C_o^{2s}(R - |x|)^{-2s} + 8r^{-2s} + \beta.$$

That is, for a suitable $C_6 > 0$,

$$(3.19) \quad 1 + w(x) \leq C_6(R - |x|)^{-2s} \quad \text{for any } x \in B_R \setminus B_{R/2}.$$

By (3.18) and (3.19), we obtain that

$$(3.20) \quad 1 + w(x) \leq C_7(R - |x|)^{-2s} \quad \text{for any } x \in B_R,$$

for a suitable $C_7 \geq 1$.

Now, we claim that

$$(3.21) \quad 1 + w(x) \leq (2^{2s} + 2^{1-2s}) C_7 (R + 1 - |x|)^{-2s} \quad \text{for any } x \in B_R.$$

Indeed, if $|x| \leq R - 1$, we have that $R - |x| \geq (R + 1 - |x|)/2$, therefore (3.21) is a consequence of (3.20). If, on the other hand, $|x| \geq R - 1$, we have that $R + 1 - |x| \leq 2$, thus we use (3.17) to obtain

$$1 + w(x) \leq 2 = 2^{1-2s} 2^{-2s} \leq 2^{1-2s} (R + 1 - |x|)^{-2s},$$

which gives (3.21).

Then, (3.21) implies the upper bound in (3.5), and the lower bound may be obtained analogously, using (3.6).

Moreover, recalling (3.15), for any $x \in B_R$,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{w(y) - w(x)}{|x - y|^{n+2s}} dy &= (2 - \beta) C_o^{-2s} \int_{\mathbb{R}^n} \frac{v(y) - v(x/C_o)}{|(x/C_o) - y|^{n+2s}} dy \\ &\leq (2 - \beta) C_o^{-2s} C_5 \left(v(x/C_o) + 16r^{-2s} \right) \\ &\leq C_o^{-2s} C_5 \left((2 - \beta) v(x/C_o) + 32r^{-2s} \right) \\ &= \tau(1 + w(x)). \end{aligned}$$

This proves (3.4) and it completes the proof of Lemma 3.1. \square

Now, we give an elementary, general estimate:

Lemma 3.2. *Let $\sigma, \mu \in (0, +\infty)$, $\nu \in (\sigma, +\infty)$ and $\gamma, R_o, C \in (1, +\infty)$.*

Let $V : (0, +\infty) \rightarrow (0, +\infty)$ be a nondecreasing function. For any $r \in [R_o, +\infty)$, let

$$\alpha(r) := \min \left\{ 1, \frac{\log V(r)}{\log r} \right\}.$$

Suppose that

$$(3.22) \quad V(R_o) \geq \mu$$

and

$$(3.23) \quad r^\sigma \alpha(r) V(r)^{(\nu-\sigma)/\nu} \leq CV(\gamma r), \quad \text{for any } r \in [R_o, +\infty).$$

Then, there exist $c \in (0, 1)$ and $R_\star \in [R_o, +\infty)$, possibly depending on μ, ν, γ, R_o and C , such that

$$V(r) \geq cr^\nu, \quad \text{for any } r \in [R_\star, +\infty).$$

Proof. Let j_1 be the smallest natural number for which $\gamma^{j_1} \geq R_o$. Notice that such a definition is well posed since $\gamma > 1$.

Let

$$(3.24) \quad c := \min \left\{ \frac{\mu}{\gamma^{\nu j_1}}, \left(\frac{1}{C\gamma^\nu} \right)^{\nu/\sigma}, \left(\frac{\nu}{2C\gamma^\nu} \right)^{\nu/\sigma} \right\}.$$

Let j_2 be the smallest integer for which

$$(3.25) \quad \frac{|\log c|}{j_2 \log \gamma} \leq \frac{\nu}{2}.$$

Let $j_\star := j_1 + j_2$. For any $j \in \mathbb{N} \cap [j_\star, +\infty)$, we define $v_j := V(\gamma^j)$. We claim that

$$(3.26) \quad v_j \geq c\gamma^{\nu j}, \quad \text{for any } r \in [R_o, +\infty).$$

The proof of (3.26) is by induction over j . First of all,

$$v_{j_\star} \geq V(\gamma^{j_1}) \geq V(R_o) \geq \mu \geq c\gamma^{\nu j_1},$$

thanks to (3.24).

Then, we suppose that (3.26) holds for some $j \geq j_\star$ and we prove it for $j+1$, via the following argument.

Since we assumed that (3.26) holds j ,

$$\begin{aligned} \alpha(\gamma^j) &= \min \left\{ 1, \frac{\log(v_j)}{\log \gamma^j} \right\} \\ &\geq \min \left\{ 1, \frac{\log(c\gamma^{\nu j})}{\log \gamma^j} \right\} \\ &= \min \left\{ 1, \nu - \frac{|\log c|}{j \log \gamma} \right\} \\ &\geq \min \left\{ 1, \frac{\nu}{2} \right\}, \end{aligned}$$

thanks to (3.25).

Therefore, using (3.23) with $r := \gamma^j$ and the assumption that (3.26) holds for j , we conclude that

$$\begin{aligned} v_{j+1} &= V(\gamma^{j+1}) \\ &\geq \frac{\gamma^{\sigma j}}{C} \alpha(\gamma^j) v_j^{(\nu-\sigma)/\nu} \\ &\geq \min \left\{ \frac{1}{C}, \frac{\nu}{2C} \right\} c^{(\nu-\sigma)/\nu} \gamma^{\nu j}. \end{aligned}$$

Recalling (3.24), we see that this last quantity is greater or equal than $c\gamma^{(j+1)\nu}$, thus completing the induction argument which proves (3.26).

From (3.26), the desired result plainly follows. \square

Remark 3.3. In the sequel, we will use Lemma 3.2 with $V(R) := |\{u > \theta_\star\} \cap B_R|$, with $\theta_\star \leq \theta_1$. In this way, condition (3.22) is warranted by (3.1).

Now, we make some useful computations, valid for any $s \in (0, 1)$. We fix $K \geq 2(R_o + 1)$, to be taken suitably large in the sequel, where R_o is the one given by the statement of Theorem 1.4, and $R > 2K$. Given any measurable $w : \mathbb{R}^n \rightarrow [-1, 1]$ such that

$$(3.27) \quad w = 1 \text{ in } \mathcal{C}B_R,$$

we define

$$(3.28) \quad v(x) := \min\{u(x), w(x)\}$$

and $D := (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}B_R \times \mathcal{C}B_R)$. Notice that

$$\mathcal{K}(u; B_R) = \frac{1}{2} \iint_D \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy$$

and that

$$(3.29) \quad v = u \text{ in } \mathcal{C}B_R.$$

So, by a simple algebraic computation, we have that

$$\begin{aligned}
& \mathcal{K}(u-v; B_R) + \mathcal{K}(v; B_R) - \mathcal{K}(u; B_R) \\
&= - \iint_D \frac{((u-v)(x) - (u-v)(y)) (v(y) - v(x))}{|x-y|^{n+2s}} dx dy \\
&= - \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{((u-v)(x) - (u-v)(y)) (v(y) - v(x))}{|x-y|^{n+2s}} dx dy \\
&= -2 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - v(x)) (v(y) - v(x))}{|x-y|^{n+2s}} dx dy \\
&= 2 \int_{\{u>v=w\}} (u(x) - v(x)) \left[\int_{\mathbb{R}^n} \frac{v(y) - w(x)}{|x-y|^{n+2s}} dy \right] dx \\
&\leq 2 \int_{B_R \cap \{u>w\}} (u(x) - v(x)) \left[\int_{\mathbb{R}^n} \frac{w(y) - w(x)}{|x-y|^{n+2s}} dy \right] dx \\
&= -2 \int_{B_R \cap \{u>w\}} (u(x) - v(x)) (-\Delta)^s w(x) dx.
\end{aligned}$$

As a consequence, using once more (3.29) and the minimality of u , we conclude that

$$\begin{aligned}
& \mathcal{K}(u-v; B_R) \\
&\leq \mathcal{E}_R(u) - \mathcal{E}_R(v) + \int_{B_R} W(v) - W(u) dx \\
&\quad - 2 \int_{B_R \cap \{u>w\}} (u(x) - v(x)) (-\Delta)^s w(x) dx \\
&\leq \int_{B_R \cap \{u>w\}} W(w) - W(u) dx \\
&\quad - 2 \int_{B_R \cap \{u>w\}} (u(x) - w(x)) (-\Delta)^s w(x) dx.
\end{aligned} \tag{3.30}$$

Now, we fix $\theta_1, \theta_2 \in (-1, 1)$ as in the statement of Theorem 1.4, and we take

$$\theta_* \leq \min\{\theta_1, \theta_2, -1+c\}, \tag{3.31}$$

with c as in (1.8). We define

$$A(R) := c \int_{B_R \cap \{w < u \leq \theta_*\}} (u-w)^2 dx. \tag{3.32}$$

From the behavior of W near its minima (see (1.8)), we deduce that

$$\begin{aligned}
& \int_{B_R \cap \{u>w\}} W(w) - W(u) dx \\
&\leq -c \int_{B_R \cap \{w < u \leq \theta_*\}} (1+w)(u-w) dx \\
&\quad + \frac{1}{c} \int_{B_R \cap \{u > \max\{w, \theta_*\}\}} (1+w) dx - A(R).
\end{aligned}$$

This and (3.30) give that

$$\begin{aligned}
 \mathcal{K}(u - v; B_R) &\leq -c \int_{B_R \cap \{w < u \leq \theta_*\}} (1 + w)(u - w) dx \\
 &\quad + \frac{1}{c} \int_{B_R \cap \{u > \max\{w, \theta_*\}\}} (1 + w) dx \\
 &\quad - 2 \int_{B_R \cap \{u > w\}} (u(x) - w(x)) (-\Delta)^s w(x) dx \\
 &\quad - A(R).
 \end{aligned}
 \tag{3.33}$$

While (3.33) is valid for any w satisfying (3.27), we now choose w in a convenient way. That is, we define $\tau := c/4$ and we take w to be the function constructed in Lemma 3.1. With this choice, (3.33) and Lemma 3.1 give that

$$\begin{aligned}
 \mathcal{K}(u - v; B_R) &+ \frac{c}{2} \int_{B_R \cap \{w < u \leq \theta_*\}} (1 + w)(u - w) dx \\
 &\leq -\frac{c}{2} \int_{B_R \cap \{w < u \leq \theta_*\}} (1 + w)(u - w) dx \\
 &\quad + \frac{1}{c} \int_{B_R \cap \{u > \max\{w, \theta_*\}\}} (1 + w) dx \\
 &\quad + 2\tau \int_{B_R \cap \{u > w\}} (u(x) - w(x)) (1 + w(x)) dx - A(R) \\
 &\leq \frac{1}{c} \int_{B_R \cap \{u > \max\{w, \theta_*\}\}} (1 + w) dx - A(R) \\
 &\leq C \int_{B_R \cap \{u > \theta_*\}} (R + 1 - |x|)^{-2s} dx - A(R),
 \end{aligned}
 \tag{3.34}$$

for a suitable $C > 0$.

Now, we define

$$V(R) := |\{u > \theta_*\} \cap B_R|.$$

Hence, using the Coarea formula, we deduce from (3.34) that

$$\begin{aligned}
 A(R) + \mathcal{K}(u - v; B_R) &+ \frac{c}{2} \int_{B_R \cap \{w < u \leq \theta_*\}} (1 + w)(u - w) dx \\
 &\leq C \int_0^R (R + 1 - t)^{-2s} \left(\int_{\partial B_t} \chi_{\{u > \theta_*\}}(x) d\mathcal{H}^{n-1}(x) \right) dt \\
 &= C \int_0^R (R + 1 - t)^{-2s} V'(t) dt.
 \end{aligned}
 \tag{3.36}$$

3.2. Completion of the proof of Theorem 1.4. Now we use the estimates of Theorem 1.6. For convenience, the proof of Theorem 1.6 itself is postponed to Section 4.

Given a measurable set $A \subseteq \mathbb{R}^n$, we define

$$\ell(A) := \begin{cases} |A|^{(1-2s)/n} & \text{if } s \in (0, 1/2), \\ \log |A| & \text{if } s = 1/2, \\ 1 & \text{if } s \in (1/2, 1). \end{cases}
 \tag{3.37}$$

Given $s \in (1/2, 1)$ and $\alpha \geq 0$, we notice that the map

$$(0, +\infty) \ni t \mapsto \alpha t^{1-2s} + t$$

has minimum at $t = (\alpha(2s-1))^{1/(2s)}$ and therefore

$$(3.38) \quad \inf_{t \in (0, +\infty)} \alpha t^{1-2s} + t \geq c_1 \alpha^{1/(2s)},$$

for a suitable $c_1 > 0$, as long as $s \in (1/2, 1)$.

Also, the map

$$(0, +\infty) \ni t \mapsto \alpha^{(n-1)/n} \log \frac{\alpha}{t} + t$$

has minimum at $t = \alpha^{(n-1)/n}$, so

$$(3.39) \quad \inf_{t \in (0, +\infty)} \alpha^{(n-1)/n} \log \frac{\alpha}{t} + t \geq \frac{\alpha^{(n-1)/n}}{n} \log \alpha.$$

Moreover, if, given $\kappa > 0$, we consider the map

$$[\kappa, +\infty) \ni t \mapsto \Phi(t) := \frac{t^{1/n}}{1 + |\log t|},$$

we have that $\Phi(t) > 0$ for any $t \in [\kappa, +\infty)$ and

$$\lim_{t \rightarrow +\infty} \Phi(t) = +\infty,$$

therefore

$$(3.40) \quad i_\kappa := \inf_{t \in [\kappa, +\infty)} \Phi(t) > 0.$$

Now, we claim that, if A and B are disjoint measurable subsets of \mathbb{R}^n and $D := \mathcal{C}(A \cup B)$, with $|A| \geq \kappa > 0$, then there exists $c_0 > 0$, possibly depending on κ such that

$$(3.41) \quad L(A, D) + |B| \geq c_0 |A|^{(n-1)/n} \ell(A).$$

To prove (3.41), we take c as⁴ in Theorem 1.6 and we distinguish two cases. First, if $|B| > c|A|$, we use (3.40) to see that

$$\begin{aligned} L(A, D) + |B| &\geq |B| > c|A| \\ &= \begin{cases} c|A|^{2s/n} |A|^{(n-2s)/n} \geq c\kappa^{2s/n} |A|^{(n-2s)/n}, & \text{if } s \in (0, 1/2), \\ c|A|^{1/n} |A|^{(n-1)/n} \geq c i_\kappa (1 + |\log |A||) |A|^{(n-1)/n}, & \text{if } s \in [1/2, 1), \end{cases} \end{aligned}$$

⁴Of course, no confusion should arise with the c in (1.8) which was previously used in (3.31) – in any case, one could just take c to be the smallest of these two constants to make them equal.

which gives (3.41). Therefore, we may suppose that $|B| \leq c|A|$ and use (1.10), (3.38) and (3.39) to conclude that

$$\begin{aligned} \frac{1}{c} \left(L(A, D) + |B| \right) &\geq \begin{cases} |A|^{(n-2s)/n} + |B| & \text{if } s \in (0, 1/2), \\ |A|^{(n-1)/n} \log(|A|/|B|) + |B| & \text{if } s = 1/2, \\ |A|^{2s(n-1)/n} |B|^{1-2s} + |B| & \text{if } s \in (1/2, 1), \end{cases} \\ &\geq \begin{cases} |A|^{(n-2s)/n} & \text{if } s \in (0, 1/2), \\ \left(|A|^{(n-1)/n} \log |A| \right) / n & \text{if } s = 1/2, \\ c_1 \left(|A|^{2s(n-1)/n} \right)^{1/(2s)} & \text{if } s \in (1/2, 1), \end{cases} \\ &\geq c_2 |A|^{(n-1)/n} \ell(A), \end{aligned}$$

for some $c_2 > 0$, proving (3.41).

Now, we take a free parameter $K > 1$, that will be chosen conveniently large in what follows. The radius R of Theorem 1.4 will be taken larger than K . We observe that, by (3.5),

$$(3.42) \quad w \leq -1 + C(K+1)^{-2s} < -1 + \frac{1+\theta_\star}{2} \text{ in } B_{R-K},$$

as long as K is large enough possibly in dependence of θ_\star which was fixed in (3.31), and so

$$(3.43) \quad a_R := B_R \cap \left\{ u - w \geq \frac{1+\theta_\star}{4} \right\} \supseteq B_{R-K} \cap \{u > \theta_\star\}.$$

By (3.1), (3.31) and (3.43), when R is large

$$|a_R| \geq |\{u > \theta_1\} \cap B_{R_0}| \geq \mu.$$

As a consequence, we may apply (3.41) with

$$\kappa := \mu,$$

$$A := a_R,$$

$$B := b_R := B_R \cap \left\{ \frac{1+\theta_\star}{8} < u - w < \frac{1+\theta_\star}{4} \right\}$$

$$\text{and } D := d_R := \mathcal{C}(A \cup B) = \mathcal{C}B_R \cup \left(B_R \cap \left\{ u - w \leq \frac{1+\theta_\star}{8} \right\} \right).$$

We obtain that

$$L(a_R, d_R) + |b_R| \geq c_3 |a_R|^{(n-1)/n} \ell(a_R)$$

for a suitable $c_3 > 0$, possibly depending on μ .

Accordingly, recalling the notation in (3.35), (3.37) and (3.43),

$$(3.44) \quad L(a_R, d_R) + |b_R| \geq c_3 V(R-K)^{(n-1)/n} \ell_{R-K},$$

where

$$(3.45) \quad \ell_R := \ell\left(B_R \cap \{u > \theta_\star\}\right) = \begin{cases} (V(R))^{(1-2s)/n} & \text{if } s \in (0, 1/2), \\ \log(V(R)) & \text{if } s = 1/2, \\ 1 & \text{if } s \in (1/2, 1). \end{cases}$$

Notice that

$$(3.46) \quad \text{the map } R \mapsto \ell_R \text{ is nondecreasing.}$$

Now, we recall (3.28) and (3.29) to see that

$$\text{if } y \in d_R \text{ then } (u - v)(y) \leq \frac{1 + \theta_\star}{8}.$$

Therefore,

$$\begin{aligned} & \text{if } y \in d_R \text{ and } x \in a_R \text{ then} \\ & |(u - v)(x) - (u - v)(y)| \geq (u - v)(x) - (u - v)(y) \\ & \geq \frac{1 + \theta_\star}{4} - \frac{1 + \theta_\star}{8} = \frac{1 + \theta_\star}{8}. \end{aligned}$$

Recalling (1.9), this implies that

$$\begin{aligned} (3.47) \quad & 2 \mathcal{H}(u - v; B_R) \\ & \geq \int_{a_R} \int_{d_R} \frac{|(u - v)(x) - (u - v)(y)|^2}{|x - y|^{n+2s}} dx dy \\ & \geq \left(\frac{1 + \theta_\star}{8} \right)^2 L(a_R, d_R). \end{aligned}$$

Furthermore, by (3.42),

$$(3.48) \quad b_{R-K} = B_{R-K} \cap \left\{ \frac{1 + \theta_\star}{8} < u - w < \frac{1 + \theta_\star}{4} \right\} \subseteq B_{R-K} \cap \{u \leq \theta_\star\}.$$

Also, by (3.32),

$$\begin{aligned} (3.49) \quad & A(R) = c \int_{B_R \cap \{w < u \leq \theta_\star\}} (u - w)^2 dx \\ & \geq c \int_{b_R \cap \{u \leq \theta_\star\}} (u - w)^2 dx \\ & \geq c_4 |b_R \cap \{u \leq \theta_\star\}|, \end{aligned}$$

for a suitable $c_4 \in (0, 1)$.

Now, we observe that, if $t \in [R - K, R]$,

$$(R + 1 - t)^{-2s} \geq (1 + K)^{-2s}$$

and therefore, recalling the notation in (3.35),

$$\begin{aligned} V(R) - V(R - K) &= \int_{R-K}^R V'(t) dt \\ &\leq \frac{1}{c_5} \int_{R-K}^R (R + 1 - t)^{-2s} V'(t) dt \leq \frac{1}{c_5} \int_0^R (R + 1 - t)^{-2s} V'(t) dt, \end{aligned}$$

for a suitable $c_5 \in (0, 1)$, depending on K , that is now fixed once and for all.

Therefore, exploiting (3.48),

$$\begin{aligned} |b_R| &\leq |b_R \cap \{u \leq \theta_\star\}| + |b_R \cap \{u > \theta_\star\}| \\ &= |b_R \cap \{u \leq \theta_\star\}| + |(b_R \setminus b_{R-K}) \cap \{u > \theta_\star\}| \\ &\leq |b_R \cap \{u \leq \theta_\star\}| + |(B_R \setminus B_{R-K}) \cap \{u > \theta_\star\}| \\ &= |b_R \cap \{u \leq \theta_\star\}| + V(R) - V(R - K) \\ &\leq |b_R \cap \{u \leq \theta_\star\}| + \frac{1}{c_5} \int_0^R (R + 1 - t)^{-2s} V'(t) dt. \end{aligned}$$

With this, we can write (3.49) as

$$\begin{aligned} A(R) + \frac{c}{2} \int_{B_R \cap \{w < u \leq \theta_\star\}} (1+w)(u-w) dx &\geq A(R) \\ &\geq c_4 |b_R| - \frac{c_4}{c_5} \int_0^R (R+1-t)^{-2s} V'(t) dt. \end{aligned}$$

The latter estimate, (3.36) and (3.47) give that

$$\begin{aligned} &C \int_0^R (R+1-t)^{-2s} V'(t) dt \\ &\geq \mathcal{K}(u-v; B_R) + A(R) + \frac{c}{2} \int_{B_R \cap \{w < u \leq \theta_\star\}} (1+w)(u-w) dx \\ &\geq c_6 \left(L(a_R, d_R) + |b_R| \right) - \frac{c_4}{c_5} \int_0^R (R+1-t)^{-2s} V'(t) dt, \end{aligned}$$

for suitable $C \in (1, +\infty)$ and $c_6 \in (0, 1)$, possibly depending on θ_\star , that was fixed in (3.31).

Therefore, taking the last term on the other side, recalling (3.44) and possibly renaming $C \geq 1$, we conclude that⁵

$$(3.50) \quad C \int_0^R (R+1-t)^{-2s} V'(t) dt \geq \ell_{R-K} V(R-K)^{(n-1)/n}.$$

Now, we notice that, if $\rho \geq 1$,

$$(3.51) \quad \int_t^{(3/2)\rho} (R+1-t)^{-2s} dR \leq \begin{cases} (4^{1-2s}/(1-2s))\rho^{1-2s} & \text{if } s \in (0, 1/2), \\ \log((5/2)\rho) & \text{if } s = 1/2, \\ 1/(2s-1) & \text{if } s \in (1/2, 1). \end{cases}$$

Therefore, we make use of (3.46) and (3.51) in order to integrate (3.50) in $R \in [\rho, (3/2)\rho]$, with $\rho \geq 2K$, and we obtain that

$$\begin{aligned} &\rho \ell_{\rho-K} V(\rho-K)^{(n-1)/n} \\ &\leq C \int_\rho^{(3/2)\rho} \left(\int_0^R (R+1-t)^{-2s} V'(t) dt \right) dR \\ &\leq C \int_0^{(3/2)\rho} \left(\int_t^{(3/2)\rho} (R+1-t)^{-2s} dR \right) V'(t) dt \\ &\leq \begin{cases} C' \rho^{1-2s} V((3/2)\rho) & \text{if } s \in (0, 1/2), \\ C' (\log \rho) V((3/2)\rho) & \text{if } s = 1/2, \\ C' V((3/2)\rho) & \text{if } s \in (1/2, 1). \end{cases} \end{aligned}$$

for some $C, C' \geq 1$.

That is, for large r , recalling (3.45) and possibly renaming $C > 1$,

$$(3.52) \quad \begin{cases} r^{2s} V(r)^{(n-2s)/n} \leq CV(2r) & \text{if } s \in (0, 1/2), \\ r \frac{\log V(r)}{\log r} V(r)^{(n-1)/n} \leq CV(2r) & \text{if } s = 1/2, \\ r V(r)^{(n-1)/n} \leq CV(2r) & \text{if } s \in (1/2, 1). \end{cases}$$

⁵The reader may observe in (3.50) the structurally different iteration between the cases $s \in (0, 1/2)$ and $s \in [1/2, 1)$, which is encoded in ℓ_{R-K} , according to (3.45).

By (3.52) and Lemma 3.2, applied here with $\sigma := 2s$ when $s \in (0, 1/2)$, or $\sigma := 1$ when $s \in [1/2, 1)$, and $\gamma := 2$ (recall also Remark 3.3), we obtain that $V(R) \geq c_o R^n$ for large R , for a suitable $c_o \in (0, 1)$.

Therefore, recalling also (3.31),

$$(3.53) \quad \begin{aligned} & |\{u > \theta_2\} \cap B_R| + |\{\theta_* < u \leq \theta_2\} \cap B_R| \\ &= |\{u > \theta_*\} \cap B_R| = V(R) \geq c_o R^n \end{aligned}$$

for large R . On the other hand, by (1.5),

$$(3.54) \quad \begin{aligned} \frac{\overline{C}R^n}{\log R} &\geq \mathcal{E}(u, B_R) \geq \int_{\{\theta_* < u \leq \theta_2\} \cap B_R} W(u(x)) dx \\ &\geq \inf_{r \in [\theta_*, \theta_2]} W(r) |\{\theta_* < u \leq \theta_2\} \cap B_R|. \end{aligned}$$

By (3.53) and (3.54), we obtain that (1.7) holds true, and this completes the proof of Theorem 1.4.

4. PROOF OF THEOREM 1.6

Given $\xi \in S^{n-1}$, we denote by π_ξ the hyperplane normal to ξ passing through the origin, namely

$$\pi_\xi := \{x \in \mathbb{R}^n \text{ s.t. } \xi \cdot x = 0\}.$$

Given $\Omega \subseteq \mathbb{R}^n$, we consider the projection of Ω along ξ , i.e.

$$\Pi_\xi(\Omega) := \{p \in \pi_\xi \text{ s.t. there exists } t \in \mathbb{R} \text{ for which } p + t\xi \in \Omega\}.$$

Next result relates the n -dimensional measure of Ω with the largest possible $(n-1)$ -dimensional measure of $\Pi_\xi(\Omega)$ (i.e., pictorially, the measure of an object in a room with the measure of its shadows on the walls and on the floor).

Lemma 4.1. *Let Ω be a measurable subset of \mathbb{R}^n . Then,*

- (i) $|\Omega|^{n-1} \leq |\Pi_{e_1}(\Omega)| \dots |\Pi_{e_n}(\Omega)|$,
- (ii) *there exists $k \in \{1, \dots, n\}$ for which*

$$|\Pi_{e_k}(\Omega)| \geq |\Omega|^{(n-1)/n}.$$

Proof. First of all, we use the generalized Hölder inequality (see, e.g., page 623 of [10]) to observe that, if $\psi_o \geq 0$ and $\psi_1, \dots, \psi_{n-1} \in L^1(\mathbb{R}, [0, +\infty])$, then

$$(4.1) \quad \begin{aligned} & \int_{\mathbb{R}} \left(\psi_o \psi_1(t) \dots \psi_{n-1}(t) \right)^{1/(n-1)} dt \\ &= \psi_o^{1/(n-1)} \int_{\mathbb{R}} \left(\psi_1(t) \dots \psi_{n-1}(t) \right)^{1/(n-1)} dt \\ &\leq \psi_o^{1/(n-1)} \left(\int_{\mathbb{R}} \psi_1(t) dt \right)^{1/(n-1)} \dots \left(\int_{\mathbb{R}} \psi_{n-1}(t) dt \right)^{1/(n-1)} \\ &= \left(\psi_o \int_{\mathbb{R}} \psi_1(t) dt \dots \int_{\mathbb{R}} \psi_{n-1}(t) dt \right)^{1/(n-1)}. \end{aligned}$$

Now, we introduce some notation. Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, for any $i \in \{1, \dots, n\}$ we define

$$\hat{x}_i := x - x_i e_i = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n).$$

Also, for any $i \leq k \in \{1, \dots, n\}$, we set

$$\hat{x}_{i;k} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k).$$

Notice that $\hat{x}_i \in \mathbb{R}^n$, $\hat{x}_{i;k} \in \mathbb{R}^{k-1}$ and the above notation means that $\hat{x}_{k;k} := (x_1, \dots, x_{k-1})$. We also stress the fact that

$$(4.2) \quad \hat{x}_i \text{ does not depend on } x_i.$$

For short, we also set $\chi := \chi_\Omega$ and $\chi_i := \chi_{\Pi_{e_i}(\Omega)}$. Then, we have that

$$(4.3) \quad \int_{\mathbb{R}^{n-1}} \chi_i(\hat{x}_i) d\hat{x}_{i;n} = |\Pi_{e_i}(\Omega)|.$$

Now, we observe that

$$\chi(x) \leq \chi_1(\hat{x}_1) \dots \chi_n(\hat{x}_n)$$

and so

$$\chi(x) = \left(\chi(x) \right)^{1/(n-1)} \leq \left(\chi_1(\hat{x}_1) \dots \chi_n(\hat{x}_n) \right)^{1/(n-1)}.$$

Hence, integrating in dx_1 and using (4.1) and (4.2),

$$\int_{\mathbb{R}} \chi(x) dx_1 \leq \left(\chi_1(\hat{x}_1) \int_{\mathbb{R}} \chi_2(\hat{x}_2) dx_1 \dots \int_{\mathbb{R}} \chi_n(\hat{x}_n) dx_1 \right)^{1/(n-1)}.$$

So, integrating in dx_2 and using (4.1) and (4.2) once more,

$$\begin{aligned} & \int_{\mathbb{R}^2} \chi(x) d(x_1, x_2) \\ & \leq \left(\int_{\mathbb{R}} \chi_1(\hat{x}_1) dx_2 \int_{\mathbb{R}} \chi_2(\hat{x}_2) dx_1 \right. \\ & \quad \left. \int_{\mathbb{R}^2} \chi_3(\hat{x}_3) d(x_1, x_2) \dots \int_{\mathbb{R}^2} \chi_n(\hat{x}_n) d(x_1, x_2) \right)^{1/(n-1)} \\ & = \left(\int_{\mathbb{R}} \chi_1(\hat{x}_1) d\hat{x}_{1;2} \int_{\mathbb{R}} \chi_2(\hat{x}_2) d\hat{x}_{2;2} \right. \\ & \quad \left. \int_{\mathbb{R}^2} \chi_3(\hat{x}_3) d(x_1, x_2) \dots \int_{\mathbb{R}^2} \chi_n(\hat{x}_n) d(x_1, x_2) \right)^{1/(n-1)}, \end{aligned}$$

where we denoted by $d(x_1, x_2)$ the volume element in \mathbb{R}^2 . By iterating this argument, for any $k \leq n$, we conclude that

$$\begin{aligned} & \int_{\mathbb{R}^k} \chi(x) d(x_1, \dots, x_k) \\ & \leq \left(\int_{\mathbb{R}^{k-1}} \chi_1(\hat{x}_1) d\hat{x}_{1;k} \dots \int_{\mathbb{R}^{k-1}} \chi_k(\hat{x}_k) d\hat{x}_{k;k} \right. \\ & \quad \left. \int_{\mathbb{R}^k} \chi_{k+1}(\hat{x}_{k+1}) d(x_1, \dots, x_k) \dots \int_{\mathbb{R}^k} \chi_n(\hat{x}_n) d(x_1, \dots, x_k) \right)^{1/(n-1)}, \end{aligned}$$

and, finally,

$$\int_{\mathbb{R}^n} \chi(x) dx \leq \left(\int_{\mathbb{R}^{n-1}} \chi_1(\hat{x}_1) d\hat{x}_{1;n} \dots \int_{\mathbb{R}^{n-1}} \chi_n(\hat{x}_n) d\hat{x}_{n;n} \right)^{1/(n-1)}.$$

This and (4.3) imply the claim in (i). Then, (ii) easily follows from (i). \square

As a curiosity, we remark that the estimates in Lemma 4.1 are optimal, as the example of the cube shows, and that they may be seen as suitably refined versions of the classical isoperimetric and isodiametric inequalities.

The main estimate needed for the proof of Theorem 1.6 is the following:

Lemma 4.2. *Let $s \in (0, 1)$. Let A and B be disjoint measurable subsets of \mathbb{R}^n , with $|A| = 1$. Let $D := \mathcal{C}(A \cup B)$. Then, there exists $\delta \in (0, 1/10)$ depending on n and s such that the following holds: if $|B| \leq \delta$, then*

$$L(A, D) \geq \begin{cases} \delta & \text{if } s \in (0, 1/2), \\ \delta \log(1/|B|) & \text{if } s = 1/2, \\ \delta |B|^{1-2s} & \text{if } s \in (1/2, 1). \end{cases}$$

Proof. The main step of the proof consists in the following estimate: there exists $\tilde{c} \in (0, 1)$, suitably small, depending on n and s , such that, for any

$$(4.4) \quad r \in [\tilde{C}|B|, \tilde{c}],$$

with $\tilde{C} := 1/\tilde{c}$, we have

$$(4.5) \quad \int_A \left[\int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx \geq \tilde{c} r^{1-2s}.$$

In order to prove (4.5), we divide \mathbb{R}^n into a collection K of nonoverlapping cubes Q of size r . We define

$$\begin{aligned} K_B &:= \left\{ Q \in K \text{ s.t. } \frac{|Q \cap B|}{|Q|} \geq \frac{1}{3} \right\}, \\ K_D &:= \left\{ Q \in K \setminus K_B \text{ s.t. } \frac{|Q \cap D|}{|Q|} \geq \frac{1}{3} \right\}, \\ K_A &:= K \setminus (K_B \cup K_D). \end{aligned}$$

We also set

$$Q_B := \bigcup_{Q \in K_B} Q, \quad Q_D := \bigcup_{Q \in K_D} Q, \quad Q_A := \bigcup_{Q \in K_A} Q.$$

We observe that

$$(4.6) \quad \delta \geq |B| \geq \sum_{Q \in K_B} |Q \cap B| \geq \frac{1}{3} \sum_{Q \in K_B} |Q| = \frac{1}{3} |Q_B|.$$

Moreover,

$$(4.7) \quad \begin{aligned} \text{if } Q \in K_A, \quad |Q \cap A| &= |Q| - |Q \cap B| - |Q \cap D| \\ &\geq |Q| - \frac{1}{3}|Q| - \frac{1}{3}|Q| = \frac{1}{3}|Q|. \end{aligned}$$

In particular,

$$(4.8) \quad |Q_A| = \sum_{Q \in K_A} |Q| \leq 3 \sum_{Q \in K_A} |Q \cap A| \leq 3|A| = 3.$$

We also point out that if $x \in Q \subset K_D$ and $\tilde{C} > \sqrt{n}$, then $Q \subseteq B_{\tilde{C}r}(x)$ and so

$$\begin{aligned}
 (4.9) \quad & \int_{Q \cap A} \left[\int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx \\
 & \geq \int_{Q \cap A} \left[\int_{(Q \cap D) \setminus B_{\tilde{c}r}(x)} \frac{dy}{|x-y|^{n+2s}} \right] dx \\
 & \geq \int_{Q \cap A} \frac{|(Q \cap D) \setminus B_{\tilde{c}r}(x)|}{(\sqrt{n}r)^{n+2s}} dx \\
 & \geq \frac{|Q \cap A| \left(r^n/3 - \tilde{c}^n |B_1| r^n \right)}{(\sqrt{n}r)^{n+2s}} \geq cr^{-2s} |Q \cap A|,
 \end{aligned}$$

provided that \tilde{c} is sufficiently small. Now, two cases may occur. Either

$$(4.10) \quad |Q_D \cap A| \geq r$$

or not. If (4.10) holds, then we exploit (4.9) to obtain

$$\begin{aligned}
 & \int_A \left[\int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx \\
 & \geq \sum_{Q \in K_D} \int_{Q \cap A} \left[\int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx \\
 & \geq \sum_{Q \in K_D} cr^{-2s} |Q \cap A| \geq cr^{-2s} |Q_D \cap A| \geq cr^{1-2s}.
 \end{aligned}$$

That is, if (4.10) holds true, then (4.5) is proved, up to renaming the constants.

Therefore, we can focus on the case in which (4.10) does not hold and suppose from now on that

$$(4.11) \quad |Q_D \cap A| < r.$$

Hence, recalling (4.4) and (4.6), and the fact that $\delta < 1/10$, we conclude that

$$\begin{aligned}
 (4.12) \quad & |Q_A| \geq |Q_A \cap A| = |A| - |Q_B \cap A| - |Q_D \cap A| \\
 & \geq 1 - |Q_B| - r \geq \frac{1}{2}.
 \end{aligned}$$

From (4.12) and Lemma 4.1(ii), we have that, up to rotation,

$$(4.13) \quad |\Pi_{e_n}(Q_A)| \geq c_0.$$

Thus, we organize the cubes of K into subfamily of columns in direction e_n : more explicitly, the column containing a cube $Q \in K$ is given by the union of all the cubes of the form $Q + j\eta e_n$, for any $j \in \mathbb{Z}$.

We define C_A to be the union of all the columns that have a cube belonging to K_A , and C_B to be the union of all the columns in C_A that have at least one cube belonging to K_B ⁶. We also let M_A and M_B to be the cardinality of the columns belonging to C_A and to C_B , respectively. We remark that

$$(4.14) \quad \text{the number of cubes in } K_A \text{ is finite,}$$

⁶Of course, C_A and C_B may well have some common columns in the intersection.

due to (4.8), and so M_A is finite⁷.

Notice that $Q_A \subseteq C_A$, therefore, by (4.13),

$$(4.15) \quad c_0 \leq |\Pi_{e_n}(C_A)| = r^{n-1} M_A.$$

On the other hand, if C_o is a column belonging to C_B , then it contains one cube $Q^{(1)}$ belonging to K_B , and therefore

$$|C_o \cap B| \geq |Q^{(1)} \cap B| \geq r^n/3.$$

Consequently,

$$|B| \geq |C_B \cap B| \geq M_B r^n/3.$$

Accordingly, recalling (4.4) and (4.15),

$$(4.16) \quad M_B \leq \frac{3|B|}{r^n} \leq \frac{3\tilde{c}r}{r^n} \leq \frac{c_0}{2r^{n-1}} \leq \frac{M_A}{2},$$

if $\tilde{c} \leq c_0/6$. As a consequence of this, using (4.15) once more, we conclude that

$$(4.17) \quad \begin{aligned} &\text{the number of columns in } C_A \setminus C_B \\ &\text{is at least } M_A - M_B \geq \frac{M_A}{2} \geq \frac{c_0}{2r^{n-1}}. \end{aligned}$$

Now, let C^* be a column in $C_A \setminus C_B$. Then, recalling (4.14), we see that C^* must contain only a finite number of cubes belonging to K_A , so we may define the cube Q_o^* as the cube of C^* belonging to C_A with the highest possible e_n -coordinate.

We consider the cube $Q_1^* := Q_o^* + 2re_n$. By construction $Q_1^* \in K_D$. Notice that if $x \in Q_o^*$ and $y \in Q_1^*$ then $|x - y| \geq r$, and $|x - y| \leq (2 + \sqrt{n})r \leq \tilde{C}r$, provided that \tilde{C} is sufficiently large. Therefore, if $x \in Q_o^*$, then $Q_1^* \subseteq B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x)$, and

$$\begin{aligned} &\int_{A \cap Q_o^*} \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dx dy}{|x - y|^{n+2s}} \\ &\geq \int_{A \cap Q_o^*} \int_{D \cap Q_1^*} \frac{dx dy}{|x - y|^{n+2s}} \\ &\geq cr^{-(n+2s)} |A \cap Q_o^*| |D \cap Q_1^*| \\ &\geq cr^{n-2s}, \end{aligned}$$

for a suitable $c \in (0, 1)$ (independent of \tilde{c} and \tilde{C} , and possibly different line after line). As a consequence,

$$\int_{C^* \cap A} \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dx dy}{|x - y|^{n+2s}} \geq cr^{n-2s}.$$

Since this is valid for any column C^* in $C_A \setminus C_B$, in the light of (4.17) we obtain that

$$\int_A \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dx dy}{|x - y|^{n+2s}} \geq cr^{1-n} r^{n-2s} = cr^{1-2s}.$$

This completes the proof of (4.5), by choosing $\tilde{c} \in (0, 1)$ small enough.

Since \tilde{c} is now fixed once and for all, we can suppose that δ in the statement of Lemma 4.2 is smaller than \tilde{c}^3 , hence $\tilde{c}^3 \geq |B|$. So, let k_0 be the largest integer so that $\tilde{c}^{2k_0+1} \geq |B|$. From (4.4) and (4.5), we deduce that, for any $1 \leq k \leq k_0$

$$\tilde{C}|B| \leq \tilde{r}_k := \tilde{c}^{2k} \leq \tilde{c},$$

⁷Similarly, exploiting (4.6), one can see that M_B is finite – however, a more precise estimate on M_B will be given in the forthcoming (4.16).

we have that

$$\int_A \left[\int_{D \cap (B_{\tilde{r}_k/\tilde{c}}(x) \setminus B_{\tilde{c}r_k}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx \geq \tilde{c}^{1+2k(1-2s)}.$$

Consequently,

$$\begin{aligned} & \int_A \left[\int_D \frac{dy}{|x-y|^{n+2s}} \right] dx \\ & \geq \sum_{k=1}^{k_0} \int_A \left[\int_{D \cap (B_{\tilde{r}_k/\tilde{c}}(x) \setminus B_{\tilde{c}r_k}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx \\ & \geq \sum_{k=1}^{k_0} \tilde{c}^{1+2k(1-2s)} \\ & \geq \begin{cases} \tilde{c}^{2(1-2s)+1} & \text{if } s \in (0, 1/2), \\ k_0 \tilde{c} & \text{if } s = 1/2, \\ \tilde{c}^{2k_0(1-2s)+1} & \text{if } s \in (1/2, 1), \end{cases} \end{aligned}$$

This implies the desired result, by taking δ appropriately small with respect to \tilde{c} . \square

We now complete the proof of Theorem 1.6 by the following argument. If $|B| > c|A|$, then $|A \cup B| \leq |A| + |B| < (1 + c^{-1})|B|$. Hence, we make use of the Sobolev-type inequality in (A.1) and we possibly rename c , to conclude that

$$\begin{aligned} (4.18) \quad L(A, D) &= \int_A \int_{\mathcal{C}(A \cup B)} \frac{dx dy}{|x-y|^{n+2s}} \\ &\geq c |A| |A \cup B|^{-2s/n} \geq c |A| |B|^{-2s/n} \end{aligned}$$

which proves (1.11).

If, on the other hand, if $|B| \leq c|A| \neq 0$, we define

$$\tilde{A} := \frac{1}{|A|^{1/n}} A, \quad \tilde{B} := \frac{1}{|A|^{1/n}} B \quad \text{and} \quad \tilde{D} := \frac{1}{|A|^{1/n}} D = \mathcal{C}(\tilde{A} \cup \tilde{B}).$$

We observe that $|\tilde{A}| = 1$ and $|\tilde{B}| = |B|/|A| \leq c$, so we can apply Lemma 4.2 and obtain that

$$\begin{aligned} L(A, D) &= |A|^{(n-2s)/n} L(\tilde{A}, \tilde{D}) \\ &\geq \begin{cases} c |A|^{(n-2s)/n} & \text{if } s \in (0, 1/2), \\ c |A|^{(n-1)/n} \log(1/|\tilde{B}|) & \text{if } s = 1/2, \\ c |A|^{(n-2s)/n} |\tilde{B}|^{1-2s} & \text{if } s \in (1/2, 1). \end{cases} \\ &= \begin{cases} c |A|^{(n-2s)/n} & \text{if } s \in (0, 1/2), \\ c |A|^{(n-1)/n} \log(|A|/|B|) & \text{if } s = 1/2, \\ c |A|^{(n-2s)/n} (|B|/|A|)^{1-2s} & \text{if } s \in (1/2, 1). \end{cases} \end{aligned}$$

This proves (1.11) and completes the proof of Theorem 1.6.

At the end of this section we prove another localized version of (1.11) in Theorem 1.6 which could be useful in other situations, such as in [21].

Proposition 4.3. *Let $s \in [1/2, 1)$. Let A, D be disjoint subsets of a cube $Q \subset \mathbb{R}^n$ with*

$$\min\{|A|, |D|\} \geq \sigma|Q|,$$

for some $\sigma > 0$. Let $B = Q \setminus (A \cup D)$. Then,

$$L(A, D) \geq \begin{cases} \delta |Q|^{\frac{n-1}{n}} \log(|Q|/|B|) & \text{if } s = 1/2, \\ \delta |Q|^{\frac{n-2s}{n}} (|Q|/|B|)^{2s-1} & \text{if } s \in (1/2, 1). \end{cases}$$

for some $\delta > 0$ depending on σ , n and s .

Proof. By rescaling, we can assume that Q is the unit cube.

The proof follows the lines of Lemma 4.2: we show that for each

$$r \in [\tilde{C}|B|, \tilde{c}],$$

we satisfy (4.5) with $\tilde{C} := 1/\tilde{c}$, for some small \tilde{c} depending on n , s and σ .

We divide the unit cube into cubes of size r that we partition into the three sets K_A , K_B and K_D . The difference now is that D is defined only on the unit cube and therefore the existence of $Q_1^* \in K_D$ at the end of the proof of Lemma 4.2 requires a more careful argument.

As before, we only need to deal with the case (see (4.12))

$$|Q_A| \geq \frac{\sigma}{2}, \quad |Q_D| \geq \frac{\sigma}{2}.$$

Denote

$$\alpha := |Q_A|, \quad \frac{\sigma}{2} \leq \alpha \leq 1 - \frac{\sigma}{2}.$$

From Lemma 4.1(ii), we have that, up to rotation,

$$(4.19) \quad |\Pi_{e_n}(Q_A)| \geq \alpha^{\frac{n-1}{n}}.$$

We define C_A to be the union of all the columns that have a cube belonging to K_A , C_B to be the columns in C_A that have at least one cube belonging to K_B and C_D to be the columns in $C_A \setminus C_B$ that have at least one cube belonging to K_D . We also let M_A , M_B , and M_D to be the cardinality of the columns belonging to C_A , C_B and to C_D , respectively.

By (4.19),

$$\alpha^{\frac{n-1}{n}} \leq |\Pi_{e_n}(C_A)| = r^{n-1} M_A.$$

and we also have (see (4.16))

$$M_B \leq 3\tilde{c}r^{1-n}.$$

On the other hand, the cardinality m_a of the columns in $C_A \setminus (C_B \cup C_D)$ (that contain only cubes from K_A) satisfies

$$m_a r^{n-1} \leq |Q_A| = \alpha.$$

Thus, if \tilde{c} is sufficiently small,

$$M_D = M_A - m_a - M_B \geq r^{1-n}(\alpha^{\frac{n-1}{n}} - \alpha - \tilde{c}/3) \geq cr^{1-n}.$$

Let C^* be a column belonging to C_D . Then

$$C^* \subset K_A \cup K_D, \quad C^* \cap K_A \neq \emptyset, \quad C^* \cap K_D \neq \emptyset,$$

and we can easily conclude that on this column there must exist two cubes $Q_0^* \subset K_A$, $Q_1^* \subset K_D$ at distance either $2r$ or $3r$ from each other. Then

$$L(A \cap Q_0^*, D \cap Q_1^*) \geq cr^{n-2s},$$

and the proof continues as before. \square

APPENDIX A. A SOBOLEV-TYPE INEQUALITY FOR SETS

For completeness, we give here an elementary proof of the Sobolev-type inequality exploited in our paper, see (4.18). For related and more general results see [4, 5] and references therein.

Lemma A.1. *Fix $x \in \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a measurable set with finite measure. Then,*

$$\int_{\mathcal{C}E} \frac{dy}{|x-y|^{n+2s}} \geq c(n, s) |E|^{-2s/n},$$

for a suitable constant $c(n, s) > 0$.

Proof. Let

$$\rho := \left(\frac{|E|}{\omega_n} \right)^{1/n}.$$

Then,

$$\begin{aligned} |\mathcal{C}E \cap B_\rho(x)| &= |B_\rho(x)| - |E \cap B_\rho(x)| \\ &= |E| - |E \cap B_\rho(x)| = |E \cap \mathcal{C}B_\rho(x)|. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathcal{C}E} \frac{dy}{|x-y|^{n+2s}} &= \int_{\mathcal{C}E \cap B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} + \int_{\mathcal{C}E \cap \mathcal{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} \\ &\geq \int_{\mathcal{C}E \cap B_\rho(x)} \frac{dy}{\rho^{n+2s}} + \int_{\mathcal{C}E \cap \mathcal{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} \\ &= \frac{|\mathcal{C}E \cap B_\rho(x)|}{\rho^{n+2s}} + \int_{\mathcal{C}E \cap \mathcal{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} \\ &= \frac{|E \cap \mathcal{C}B_\rho(x)|}{\rho^{n+2s}} + \int_{\mathcal{C}E \cap \mathcal{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} \\ &\geq \int_{E \cap \mathcal{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} + \int_{\mathcal{C}E \cap \mathcal{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} \\ &= \int_{\mathcal{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}}. \end{aligned}$$

By using polar coordinate centered at x , the desired result easily follows. \square

By integrating in x the estimate of Lemma A.1, we obtain:

Corollary A.2. *Let $E, F \subset \mathbb{R}^n$ be measurable sets with finite measure. Then,*

$$(A.1) \quad \int_F \int_{\mathcal{C}E} \frac{dx dy}{|x-y|^{n+2s}} \geq c(n, s) |F| |E|^{-2s/n},$$

for a suitable constant $c(n, s) > 0$.

In particular,

$$(A.2) \quad \int_E \int_{\mathcal{C}E} \frac{dx dy}{|x-y|^{n+2s}} \geq c(n, s) |E|^{(n-2s)/n}$$

for any measurable set E with finite measure.

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